东北数学
NORTHEAST. MATH. J.
20(4)(2004), 379–382

Estimation and Testing for the Parameters of AR(p)-ARCH(q) under Ordered Restriction*

WANG De-hui[1] (王德辉), SONG Li-xin[2] (宋立新) and SHI Ning-zhong[3] (史宁中)
(*1. School of Mathematics, Jilin University, Changchun, 130012*)
(*2. Department of Applied Mathematics, Dalian Institute of Technology, Dalian, 116023*)
(*3. Department of Mathematics, Northeast Normal University, Changchun, 130021*)

Communicated by LI Rong-hua (李荣华)

Key words: AR(p)-ARCH(q) model, ordered restriction, maximum likelihood estimator, asymptotic properties, quadratic program
1991 MR subject classification: 62M10 62F30, 62F10, 62F03
CLC number: O212.1
Document code: A
Article ID: 1000-1778(2004)04-0379-04

Suppose that the time series X_t satisfies

$$\begin{cases} X_t = \beta_1 X_{t-1} + \cdots + \beta_p X_{t-p} + \varepsilon_t, \\ \varepsilon_t = \xi_t h_t^{\frac{1}{2}}, \\ h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \cdots + \alpha_q \varepsilon_{t-q}^2, \end{cases} \tag{1}$$

where $\alpha_0 \geq \delta > 0$, $\alpha_i \geq 0$ for $i = 1, 2, \cdots, q$; β_i, $i = 1, \cdots, p$, are real numbers; p and q are the order of the model. The sequence $\{\xi_t\} \overset{\text{i.i.d}}{\sim} N(0, 1)$ and is independent of $\{h_s, s \leq t\}$ for fixed t. The above model is usually written as AR(p)-ARCH(q).

We consider stationary series AR(p)-ARCH(q) model and assume the stationary field is Θ_0. We express this statement as

$$\text{H}_1: \quad \alpha_1 \geq \alpha_2 \cdots \geq \alpha_q, \ \beta_1 \geq \beta_2 \geq \cdots \geq \beta_p$$

and we consider an order restricted testing problem, which is to test

$$\text{H}_0: \quad \alpha_1 = \alpha_2 = \cdots = \alpha_q, \ \beta_1 = \beta_2 = \cdots = \beta_p$$

against $\text{H}_1 - \text{H}_0$. We derive the likelihood ratio (LR) test statistic and its asymptotic distribution under H_0.

Let

$$\phi = (\phi_0, \phi_1, \cdots, \phi_q, \phi_{q+1}, \cdots, \phi_{q+p})' = (\alpha_0, \alpha_1, \cdots, \alpha_q, \beta_1, \beta_2, \cdots, \beta_p)'.$$

For the model (1), the log-likelihood function is

*Received date: Feb. 23, 2004.
Foundation item: The NNSF (10271049) of China.

$$L_n(\phi) = -\frac{1}{2n}\sum_{t=1}^{n}(\ln h_t(x_t;\phi) + \frac{\varepsilon_t^2}{h_t(x_t;\phi)}). \tag{2}$$

Now we introduce an algorithm to compute the restriction maximum likelihood estimators (RMLE) under H_1. Since the objective function $L_n(\phi)$ given in (2) is not convex, we use the Successive Quadratic Programming (SQP) method.

Let

$$f_n(\phi) = -L_n(\phi).$$

We consider the following optimization problem:

$$\begin{cases} \min f_n(\phi), \\ C_i(\phi) = \phi_{i+1} - \phi_i \le 0, \quad i = 1, 2, \cdots, q-1, q+1, \cdots, q+p-1, \\ C_q(\phi) = -\phi_q \le 0, \\ C_{q+p}(\phi) = \phi_1 + \phi_2 + \cdots + \phi_q - 1 \le 0, \\ C_{q+p+1}(\phi) = \delta - \phi_0 \le 0, \\ \phi \in \Theta_\beta = \{\phi;\ 1 - \phi_{q+1}z - \phi_{q+2}z^2 - \cdots - \phi_{q+p}z^p \ne 0,\ |z| < 1\}. \end{cases} \tag{3}$$

By using the SQP algorithm, we can obtain a vector series $\{\phi^{(k)}\}$ which converges to the RMLE $\hat{\phi}^* \in R^{p+q+1}$ if the condition given in the following Theorem 2 holds. Because of the parameter r and the update of $B^{(k)}$ in each iteration, the algorithm not only converges fast but also guarantees to obtain a nonnegative solution $(\hat{\phi}^*)$. Assume that $\tilde{\phi}$ is the true value of the model, we have the following theorem.

Theorem 1 (Strong consistency of RMLE) *Assume that the time series $\{X_t\}$ satisfies (1) and $\tilde{\phi} \in \Theta$. Then RMLE $\hat{\phi}^*$ is strongly consistent, that is, $Pr\{\lim_{n\to\infty} \hat{\phi}^* = \tilde{\phi}\} = 1$.*

Theorem 2[1] (Convergence of algorithm) *Assume that the sequences $\{\phi^{(k)}\}$, $\{d^{(k)}\}$, $\{v^{(k)}\}$ given in the algorithm are all bounded, and $\{B^{(k)}\}$ is a positive definite matrix sequence which is uniformly bounded, namely, there exist constants $\Lambda \ge \lambda > 0$ such that*

$$\lambda d'd \le d'B^{(k)}d \le \Lambda d'd$$

for all $d \in R^{p+q+1}$ and $k = 0, 1, 2, \cdots$. Then any cluster point ϕ^* of $\{\phi^{(k)}\}$ satisfies the Kuhn-Tucker condition of (3) when the penalizing parameter r is large enough.

We denote the non-restricted MLE by $\hat{\phi} \in R^{p+q+1}$. Weiss[2] proved asymptotic normality of $\hat{\phi}$ under the fourth moment condition. Recently, Ling and McAleer[3] proved the same result under second moment condition, that is,

$$\sqrt{n}(\hat{\phi} - \tilde{\phi}) \xrightarrow{d} N(0, I^{-1}(\tilde{\phi})), \quad I^{-1}(\tilde{\phi}) = \begin{bmatrix} I_\alpha^{-1}(\tilde{\alpha}) & 0 \\ 0 & I_\beta^{-1}(\tilde{\beta}) \end{bmatrix}. \tag{4}$$

Under the second moment condition, An and Chen[4] proved the ergodic property of stationary AR(p)-ARCH(q). As following, we consider the strong consistency of $\hat{\phi}$.

Firstly, we can obtain that for almost all ω and $n > n(\omega)$, the maximum point of $L_n(\phi)$ must belong to

$$\Theta^* = \{\phi : \delta \le \alpha_0 \le M_2,\ 0 < \phi_1 + \cdots + \phi_q < 1,\ \phi_i \ge 0,\ i = 1, \cdots, q, \\ |\phi_{q+j}| \le M_1,\ j = 1, \cdots, p\}.$$

Without loss of generality, we assume that the parameter set is $\Theta^\star$ and attain the following lemma.

Lemma 1([2], Theorem 2.1) *Assume that the time series $\{X_t\}$ follows the model* (1). *Then $\tilde{\phi}$ is the unique maximum point of $EL_n(\phi)$ for any $\phi \in \Theta_0$.*

Lemma 2 *Suppose that the time series $\{X_t\}$ follows the model* (1) *and $\alpha \in \Theta^\star$. Then as $n \longrightarrow +\infty$,*

$$I_n = \sup_{\phi} |L_n(\phi) - EL_n(\phi)| \longrightarrow 0, \qquad \text{a.s.}$$

From Lemmas 1 and 2, and Birkhoff's ergodic theorem, we can obtain

$$EL_n(\hat{\phi}) - EL_n(\tilde{\phi}) \longrightarrow 0, \qquad \text{a.s. } n \longrightarrow +\infty. \tag{5}$$

Since for any fixed ω, $\hat{\phi}$ are bounded for all n, any convergent subsequence of $\hat{\phi}$, say $\hat{\phi}^k = (\hat{\alpha}_0^k, \hat{\alpha}_1^k, \cdots, \hat{\alpha}_q^k, \hat{\beta}_1^k, \cdots, \hat{\beta}_p^k)'$, converges to $\phi^* = (\alpha_0^*, \alpha_1^*, \cdots, a_q^*, \beta_1^*, \cdots, \beta_p^*)$. Since $EL_n(\phi)$ is a continuous function of ϕ, we have

$$EL_n(\hat{\phi}^k) \longrightarrow EL_n(\phi^*), \qquad k \to +\infty. \tag{6}$$

Then by (5) and (6) and Lemma 1 we have $\phi^* = \tilde{\phi}$. Such we can attain the following theorem.

Theorem 3 *Assume that the time series $\{X_t\}$ follows* (1), *and $\tilde{\phi} \in \Theta^\star$. Then*

$$Pr\{\lim_{n\to+\infty} \hat{\phi} = \tilde{\phi}\} = 1.$$

Similarly to the Proof of Lemma 2, we can prove the following theorem.

Theorem 4 *Assume the assumptions in Theorem 3 and $\tilde{\phi} \in \Theta^{\star\circ}$, where $\Theta^{\star\circ}$ is the interior point set of $\Theta^\star$. Then as $n \to +\infty$ we have*

$$\sup_{\phi\in\Theta^{\star\circ}} \left| \frac{\partial^2 L_n(\phi)}{\partial\phi_i\partial\phi_j} - E\frac{\partial^2 L_n(\phi)}{\partial\phi_i\partial\phi_j} \right| \xrightarrow{\text{a.s.}} 0, \qquad 0 \le i,j \le p+q+1,$$

$$\left.\frac{\partial^2 L_n(\phi)}{\partial\phi\partial\phi'}\right|_{\phi=\hat{\phi}} = \left.\begin{bmatrix} \frac{\partial^2 L_n(\phi)}{\partial\alpha\partial\alpha'} & \frac{\partial^2 L_n(\phi)}{\partial\alpha\partial\beta'} \\ \frac{\partial^2 L_n(\phi)}{\partial\alpha\partial\beta'} & \frac{\partial^2 L_n(\phi)}{\partial\beta\partial\beta'} \end{bmatrix}\right|_{\phi=\hat{\phi}} \xrightarrow{\text{a.s.}} E\left.\begin{bmatrix} \frac{\partial^2 l_t(\phi)}{\partial\alpha\partial\alpha'} & \frac{\partial^2 l_t(\phi)}{\partial\alpha\partial\beta'} \\ \frac{\partial^2 l_t(\phi)}{\partial\alpha\partial\beta'} & \frac{\partial^2 l_t(\phi)}{\partial\beta\partial\beta'} \end{bmatrix}\right|_{\phi=\tilde{\phi}}$$

$$= -I(\tilde{\phi}),$$

where $I(\tilde{\phi}) = \begin{bmatrix} I_\alpha(\tilde{\alpha}) & 0 \\ 0 & I_\beta(\tilde{\beta}) \end{bmatrix}$ is the Fisher information matrix given by

$$I(\tilde{\phi}) \triangleq E\left.\left[\frac{\partial l_t(x_t;\phi)}{\partial\phi} \cdot \frac{\partial l_t(x_t;\phi)}{\partial\phi'}\right]\right|_{\phi=\tilde{\phi}}.$$

Assume that $\{X_t\}$ follows (1), and $\tilde{\phi} \in \Theta^{\star\circ}$, where $\Theta^{\star\circ}$ is the interior point set of $\Theta^\star$. And then consider the following testing problem:

$$\mathrm{H}_0 \quad \text{v.s. } \mathrm{H}_1 - \mathrm{H}_0. \tag{7}$$

For testing problem (7), the likelihood ratio test statistics is

$$\Lambda = \frac{\max_{\phi\in\mathrm{H}_0} f(x_1, x_2, \cdots, x_n; \phi \mid \mathcal{F}_0)}{\max_{\phi\in\mathrm{H}_1} f(x_1, x_2, \cdots, x_n; \phi \mid \mathcal{F}_0)}$$

and the logarithm of likelihood ratio test statistics

$$\ln \Lambda = \sum_{t=1}^{n} [l_t(x_t;\ \hat{\phi}_{H_0}) - l_t(x_t;\ \hat{\phi}^*)].$$

For the asymptotic distribution of $-2\ln\Lambda$, we get the following theorem.

Theorem 5 *Let $T = -2\ln\Lambda$. If H_0 holds, then for any $t > 0$,*

$$\lim_{n\to+\infty} P(T \geq t) = \sum_{k=1}^{q+p-2} P(k, I(\tilde{\phi}))Pr\{\chi_k^2 \geq t\},$$

where $P(k, I(\tilde{\phi}))$ is the level probability, and χ_k^2 denotes a chi-square distribution with k degrees of freedom.

In applications, $\tilde{\phi}$ is usually replaced by $\hat{\phi}$. Shi[5] used this statistic for testing the homogeneity of odds ratios, and the simulation results in his paper are satisfactory. $P(k, I(\tilde{\phi}))$ was called the level probability by Robertson and Wright[6]. It can be computed approximately from $I(\tilde{\phi})$ and the computing methods given by Kudô[7] for $k \leq 3$. For $k \leq 10$, a subroutine of computing the level probabilities was given by Bohrer and Chow[8], in which a subroutine of computing the orthant probability is needed and can be found in [9] as well.

Remark All the details of proofs of lemmas and theorems,and simulation will be published elsewhere.

References

[1] Cimu, J. X. and Fudao, Y. F., Optimization Method (in Japanese), Gongli Publishing Company, Tokyo, 1993.

[2] Weiss, A., Asymptotic theory for ARCH models: estimation and testing, *Econometric Theory,* **2**(1986), 107–131.

[3] Ling, Shiqing and Michael McAleer, Asymptotic theory for a vector ARMA-GARCH model, *Econometric Theory,* **19**(2003), 280-310.

[4] An, H. Z. and Chen, M., Nonlinear Time Series Analysis, Shanghai Science and Technology Publishing Company, Shanghai, 1998.

[5] Shi, N. Z., A test of homogereity of odds ratios again order restrictions, *J. Amer. Statist. Assoc.,* **86**(1991), 154–158.

[6] Robertson, T. and Wright, F. T., On approximation of the level probabilities and associated distributions in order restricted inference, restricted statistical inference, *Biometrika,* **70**(1983), 597–606.

[7] Kudô, A., A multivariate analogue of the one-sided test, *Biometrika,* **50**(1963), 403–418.

[8] Bohrer, R. and Chow, W., Algorithm AS 122, weights for one-sided multivariate inference, *Appl. Statist.,* **27**(1978), 100–104.

[9] Sun, H. -J., A fortran subroutine for computing normal orthant probabilities of dimensions up to nine, *Comm. Statist. Simulation Comput.,* **17**(1988), 1097–1111.